Gordan Savin talk on "Some observations about McGovern's ideal" on 07/21/10 at 2PM

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1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and fix a maximal Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. This gives rise to a set of roots Φ. Fix a set of positive roots Φ^+. For every root $\alpha \in \Phi$, we have Chevalley generators X_α, as well as $H_\alpha \in \mathfrak{h}$. The $\mathbb{Q}$-span of X_α can give you a rational form of $\mathfrak{g}$.

We want to define the Chevalley involution. This is an involution $\mathfrak{g} \to \mathfrak{g}$ which sends X_α to $-X_\alpha$, and H_α to $-H_\alpha$.

Example: For $\mathfrak{sl}_n$, $\theta(X) = -X^{tr}$.

Write $\mathfrak{g} = k \oplus \mathfrak{p}$. Let's talk about $(\mathfrak{g}, K)$-modules. Up to isogeny, $(\mathfrak{g}, K)$ corresponds to the split real form of G. Let V be a $(\mathfrak{g}, K)$-module. Let $\tau \in Irr(K)$. Consider $Hom_K(\tau, V)$, the space of intertwiners. This is naturally a U^K-module, where $U = U(\mathfrak{g})$, and we are taking invariants under the adjoint action. If V is irreducible, then $Hom_K(\tau, V)$ is irreducible U^K-module. If $\mathfrak{g}$ is $\mathfrak{sl}_2$, then U^K is abelian.

More generally, let J be the annihilator of V. You don't have to look at U^K. You can look at $(U/J)^K$. If you're lucky, if J is big enough, then $(U/J)^K$ will be abelian. If it's abelian, then V has multiplicity-free K-types.

Example: Let J be the Joseph ideal. You can decompose U/J under the adjoint action of $\mathfrak{g}$, and it decomposes as $\oplus_{n=0}^{\infty} V_{n\alpha_0}$, where α_0 is the highest root. Note that $V_{\alpha_0} \cong \mathfrak{g}$ as a $\mathfrak{g}$-module. Then,

$$(U/J)^K = \oplus_{m=0}^{\infty} V_{2m\alpha_0}^K$$

Consider the map $U(k) \to U/J$. Since J is the Joseph ideal, we get that $(U/J)^K$ is abelian. So for $(\mathfrak{g}, K)$-modules that are annihilated by the Joseph ideal, we have some nice things going on.

Example: McGovern's ideal. This ideal is $J :=$the maximal ideal with infinitesimal character $\rho/2$. This ideal naturally arises as the annihilator of some natural representation involving 2-fold central extensions and Shimura correspondence. Then under the adjoint action of $\mathfrak{g}$,

$$U/J = \oplus V$$

where the direct sum ranges over all V such that $V \cong V^*$. What Savin can show (joint paper with Loke), is that

$$(U/J)^K = U(k)^K$$

Therefore, any representation that is annihilated by J will have multiplicity-free K-types. In work of Kostant, he gives a decomposition $U(\mathfrak{g}) \cong Z(\mathfrak{g}) \otimes H$, where H denotes the harmonics. Then we are interested in when V occurs in H. Let $\mathfrak{n} = \oplus_{\alpha \in \Phi^+} \mathbb{C} X_\alpha$. Let $N = exp(\mathfrak{n})$. Then $U(\mathfrak{n})^N = Z(U(\mathfrak{n}))$. Moreover, $U(k)^K = U(\mathfrak{n})^{MN}$, where $M = Z_K(\mathfrak{h})$. Let

$$h = \sum_{\alpha \in \Phi^+} H_\alpha$$

let V be a locally finite $\mathfrak{g}$-module. Then under the adjoint action of $\mathfrak{h}$, we have $V = \oplus_{i \in \mathbb{Z}} V(i)$, where $V(i) = \{v \in V : hv = iv\}$

Lemma 1.1. *Let $v \in V^K$. Write $v = \sum v_i$ where $v_i \in V(i)$. Let v_{i_0} be the nonzero component in this sum with the largest index. Then $\mathfrak{n} v_{i_0} = 0$. In particular, if V is irreducible, then v_{i_0} is a highest weight vector.*

Thus, we can define a map

$$m : U(k)^K \to U(\mathfrak{n})^{MN}$$

It is a multiplicative map. The associated map on the graded objects, $m : gr(U(k)^K) \to gr(U(\mathfrak{n})^{MN})$, is injective. We claim that it is an isomorphism. This will follow from the result $(U/J)^K = U(k)^K$. In fact, we have $U(\mathfrak{n})^{MN} \hookrightarrow U/J$.